Seismic modeling and migration using REM with Hermite and Laguerre polynomials.

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Abstract

In this work we propose an alternative solution to the acoustic wave equation through the rapid expansion method (REM), but now using the Hermite and Laguerre polynomials instead of the Chebyshev polynomials to expand the cosine operator. This new solution can also be reduced to conventional finite-difference schemes of second and fourth order when the time step is taken relatively small. In addition, this new version of REM allows one to use a larger time-step than used in conventional finite difference schemes to march the wavefield in time generating the stable propagation of seismic waves free of numerical dispersion.
To test the efficiency of the proposed method, we apply this new rapid expansion method for modeling and reverse time migration (RTM) of synthetic data examples from a complex geological structure including a salt body. The results were satisfactory and with few dispersion and show good imaging of complex structures, thus demonstrating the effectiveness and applicability of the method.

Introduction

Finite difference (FD) is well known and a popular numerical solution for the wave equation. It has been common to use FD approximation for both the time and spatial evolution of wavefields. Although easy to solve, it is only conditionally stable which imposes a limit on the marching time step size. On the other hand, all the finite difference methods suffer from numerical dispersion problems. Various alternative approaches have been proposed in the geophysical literature to archieve stability and dispersion-free extrapolation of scalar waves in heterogeneous media for large time steps (Du et al., 2014). These methods are all based on mixed-domain space/wavenumber time extrapolation.
In the two-step extrapolation equation, the cosine term can be expanded either by a Taylor series or by orthogonal polynomials such as Chebyshev polynomials. The Chebyshev approximation for time extrapolation was introduced by Tal-Ezer (1986) and Tal-Ezer et al. (1987) and it is the basis for the rapid expansion method (REM) used by Kosloff et al. (1989); Pestana and Stoffa (2009, 2010) and Stoffa and Pestana (2009). The REM was initially developed for nonrecursive evaluation of wavefields and recently proposed by Pestana and Stoffa (2010) to be used as recursive extrapolators.
The wave-equation solution based on the REM using Chebyshev polynomial approximation is more accurate than the usual finite difference schemes. It also provides the base for a recursive solution which, in comparison with finite difference schemes, is more accurate, thus introducing less errors and leading to a stabler numerical method. Therefore, this kind of approach allows us to march in time with larger time steps.
Since the Chebyshev polynomial is the most frequently used orthogonal polynomial in most numerical approximation theory, other kinds of orthogonal polynomials should also be applicable for time evolution problems based on the wave equation solution. The argument of the Chebyshev polynomial is bounded to the interval $[-1;1]$, a feature that is not present in the case of Hermite and Laguerre polynomials. So, expansion in terms of these kinds of orthogonal polynomials may have some advantages and we will explore the efficiency and accuracy of the two-step wave equation solution based on these orthogonal polynomials.
In this paper we construct the approximation of the cosine operator using the Hermite and Laguerre polynomials and found that these orthogonal polynomials do have the required properties. Unlike the REM, it does not suffer from numerical stability or numerical dispersion. Therefore, we can explore it to design a cost-effective and high quality method for modeling and migration of seismic data.

Theory

We consider the following acoustic wave equation:

$$\frac{\partial^2 u(\mathbf{x},t)}{\partial t^2} + L^2 u(\mathbf{x},t) = f(\mathbf{x},t) \tag{1}$$

where $-L^2 = c^2(\mathbf{x})\nabla^2$, $v(\mathbf{x})$ is the velocity of propagation, $\mathbf{x} = (x,y,z)$ is the position vector and $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ is the Laplacian operator in Cartesian coordinates and $f(\mathbf{x},t)$ is the source term.

The approach that we use to solve equation 1 is called variations of parameters (VOP). Thus, the general solution of $u(\mathbf{x},t)$ to equation (1) on $[0,t]$ is written as:

$$P(\mathbf{x},t) = P_0\cos(Lt) + \frac{\dot{P}_0}{L}sin(Lt) + \frac{1}{L}\int_0^t f(\mathbf{x},s)\sin[L(t-s)]ds \tag{2}$$

where $P(\mathbf{x},t=0) = P_0$ and $\frac{\partial P(\mathbf{x},t)}{\partial t}\,|_{t=0} = \dot{P}_0$.

Equation 2 is the fundamental equation from which we derive the integration procedure. Now, if equation 2 is reevaluated using the intervals $[t, t+\Delta t]$ and $[t, t-\Delta t]$ and by adding them and evaluating the resulting integral, we obtain the following complete solution of 1, including the source term, which is given by:

$$P(\mathbf{x},t+\Delta t) + P(\mathbf{x},t-\Delta t) = 2\cos(L\Delta t)P(\mathbf{x},t) + S(\mathbf{x},t\pm\Delta t) \tag{3}$$

where $S(\mathbf{x},t\pm\Delta t)=\frac{\Delta t^2}{2}[f(\mathbf{x},t+\Delta t)+f(\mathbf{x},t-\Delta t)]$.

Rapid Expansion Method

Now following Kosloff et al. (1989) and based on the expansion method presented by Tal-Ezer et al. (1987) , the cosine function can be expanded in the following form:

$$\cos(L\Delta t)=\sum_{k=0}^{\infty}C_{2k}J_{2k}(\Delta tR)Q_{2k}\left(\frac{iL}{R}\right), \tag{4}$$

where $C_{2k}=1$ for $k=0$ and $C_{2k}=2$ for $k>0$, J_{2k} represents the Bessel function of order $2k$ and $Q_{2k}(w)$ are the modified Chebyshev polynomials. The term R is a scalar larger than the range eigenvalues of $-L^2$ and it is the same R which appeared in the original Tal-Ezer method (Tal-Ezer et al., 1987).

Since 4 contains only even polynomials, it is more convenient to use the relation,

$$Q_{k+2}(w)=2(1+2w^2)Q_k(w)-Q_{k-2}(w). \tag{5}$$

The recursion is initiated by

$$Q_0(w)=1 \quad \text{and} \quad Q_2(w)=1+2w^2, \tag{6}$$

where we have replaced w by iL/R.

For 2D wave propagation, and considering the constant velocity case, the value of R is given by $R=\pi c\sqrt{(1/\Delta x^2)+(1/\Delta z^2)}$. But, in general, c should be replaced by c_{max}, the highest velocity in the grid, and Δx, Δy and Δz are the spatial grid spacing (Tal-Ezer et al., 1987).

The sum in 4 is known to converge exponentially for $k>\Delta tR$ and, therefore, the summation can be safely truncated with a k value slightly greater than ΔtR.

Hermite polynomial expansion

In order to obtain the expansion in terms of Hermite polynomials, we start from its generating function (Arfken, 1985)

$$e^{-s^2+2sx}=\sum_{k=0}^{\infty}\frac{s^k}{k!}H_k(x), \tag{7}$$

where H_k denotes the Hermite polynomials of order k. The exponential operator can be rearranged as

$$e^{-iL\Delta t}=e^{-(\Delta t/2\lambda)^2}e^{-(-i\Delta t/2\lambda)^2+2\lambda L(-i\Delta t/2\lambda)}, \tag{8}$$

here an arbitrary parameter λ was introduced for convenience of later use. By comparing 8 with 7 by setting $s=-i\Delta t/2\lambda$ and $x=\lambda L$, we immediately obtain the Hermite expansion for the exponential operator as:

$$e^{-iL\Delta t}=e^{-(\Delta t/2\lambda)^2}\sum_{k=0}^{\infty}\frac{(-i)^k}{k!}\left(\frac{\Delta t}{2\lambda}\right)^k H_k(\lambda L). \tag{9}$$

About the convergence of this series, it is easily shown that when $k>(e\Delta t/2\lambda)$ the term $(1/k!)(\Delta t/2\lambda)^k$ will behave like $e^{-kln[k/e(2\lambda/t)]}$, which means that the expansion converges exponentially (Hu, 1999).

When k is an odd number, $H_k(-x)=-H_k(x)$. Thus, every odd term will be cancelled, and we get the following expansion

$$cos(L\Delta t)=e^{-(\Delta t/2\lambda)^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k!}\left(\frac{\Delta t}{2\lambda}\right)^{2k}H_{2k}(\lambda L). \tag{10}$$

The series of Hermite polynomials of λL can be calculated by the following recursion relation:

$$H_{k+1}(\lambda L)=2\lambda LH_k(\lambda L)\,-\,2kH_{k-1}(\lambda L), \tag{11}$$

with the initial values $H_0(\lambda L)=1$ and $H_1(\lambda L)=2\lambda L$.

Using the solution given by 3, and the Hermite expansion for $\cos(L\Delta t)$, we have that:

$$P(\mathbf{x},t+\Delta t)\,+\,P(\mathbf{x},t-\Delta t)=2\left[e^{-(\Delta t/2\lambda)^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k!}\left(\frac{\Delta t}{2\lambda}\right)^{2k}H_{2k}(\lambda L)\right]P(\mathbf{x},t)+S(x,t\pm\Delta t) \tag{12}$$

Now we need a recursion for H_k with only even terms. Using recursion 11, we get

$$H_{k+2}(x)=(4x^2-4k-2)\,H_k(x)\,-\,4k(k-1)\,H_{k-2}(x), \tag{13}$$

for $k>2$, with the initial values $H_0=1$ and $H_2=4x^2-2$.

If we consider in the Hermite expansion only the first two terms, we get the following result:

$$P(\mathbf{x},t+\Delta t)+P(\mathbf{x},t-\Delta t)=2Z\left[1+\frac{1}{2}\left(\frac{\Delta t}{2\lambda}\right)^2 (4(\lambda L)^2-2)\right]P(\mathbf{x},t)\,+\,S(x,t\pm\Delta t) \tag{14}$$

where $Z=e^{-(\Delta t/2\lambda)^2}$ or $Z=1-(\Delta t/2\lambda)^2+\frac{(\Delta t/2\lambda)^4}{2!}+...$ (series Taylor approximation).

For a sufficient small Δt, we have that $(\Delta t/2\lambda)^2\approx 0$ and $Z=1$, resulting in:

$$P(\mathbf{x},t+\Delta t)+P(x,t-\Delta t)=2P(\mathbf{x},t)-\Delta t^2L^2P(\mathbf{x},t)+S(\mathbf{x},t\pm\Delta t) \tag{15}$$

That is the scheme obtained for the wave equation when it is solved by the pseudo-spectral method (2nd order FD in time and Fourier method for spatial derivatives).

Expansion using Laguerre polynomials

The Laguerre polynomials are orthogonal, as are the Hermite, with respect to the weight function x^me^{-x}. Now further using the following relation between Hermite and Laguerre polynomials,

$$H_{2k}(x)=(-1)^k\,2^{2k}\,k!\,\mathscr{L}_k^{-1/2}(x^2), \tag{16}$$

we obtain the following result to $\cos(L\Delta t)$, as

$$\cos(L\Delta t)=e^{-(\Delta t/2\lambda)^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k!}\left(\frac{\Delta t}{2\lambda}\right)^{2k}H_{2k}(\lambda L) =\sum_{k=0}^{\infty}C_k\,(\lambda\Delta t)\,\phi_k(\lambda^2L^2); \tag{17}$$

where

$$C_k(\lambda\Delta t)=e^{-(\Delta t/2\lambda)^2}\frac{k!\,2^{2k}}{2k!}\left(\frac{\Delta t}{2\lambda}\right)^{2k}. \tag{18}$$

The expansion coefficient $C_k(\lambda\Delta t)$ can be calculated through its recurrence relation:

$$C_k(\lambda\Delta t)=\left[\frac{2}{2k-1}\right]\left(\frac{\Delta t}{2\lambda}\right)^2C_{k-1}(\lambda\Delta t) \tag{19}$$

and,

$$\phi_k(\lambda^2 L^2) = \mathscr{L}_k^{-1/2}(\lambda^2 L^2) \quad (20)$$

which satisfy the recurrence relation:

$$\phi_k(x^2) = \frac{(2k-3/2-x^2)}{k}\phi_{k-1}(x^2) - \frac{(k-3/2)}{k}\phi_{k-2}(x^2) \quad (21)$$

with the initial values:

$$\phi_0(x^2) = 1 \quad \text{and} \quad \phi_1(x^2) = (1/2 - x^2)$$

where we have replacede x by (λL)

Now, using the wave equation solution 3 and replacing the $\cos(L\Delta t)$ by the new approximation, we can produce the following recursive solution:

$$P(\mathbf{x}, t+\Delta t) + P(\mathbf{x}, t-\Delta t) = 2\sum_{k=0}^{\infty} C_k(\lambda\Delta t)\phi_k(\lambda^2 L^2)\, P(\mathbf{x}, t) + S(\mathbf{x}, t \pm \Delta t) \quad (22)$$

Using only two recursion terms (2nd order approximation in time), we obtain:

$$P(\mathbf{x}, t+\Delta t) + P(\mathbf{x}, t-\Delta t) = 2\alpha P(\mathbf{x}, t) - \beta\Delta t^2 L^2 P(\mathbf{x}, t) + S(\mathbf{x}, t \pm \Delta t) \quad (23)$$

where $\alpha = C_0[1 + (\Delta t/2\lambda)^2]$ and $\beta = C_0 = e^{-(\Delta t/2\lambda)^2}$.

Numerical Results

To compare the different series expansions to the cosine function designed in the paper, in Figure 1, we have the expansion of $\cos(\phi)$ using the Taylor series, Chebyshev polynomials and by the Laguerre polynomials. The ϕ is defined as $\phi = L\Delta t$ and $\phi_{max} = R\Delta t$, where R was determined previously. We consider the 2-D case and $\Delta x = \Delta z = 0.012\ km$, $c_{max} = 4,480\ km/s$, $\Delta t = 2\ ms$, then $\phi_{max} = 3.31\ rad$.

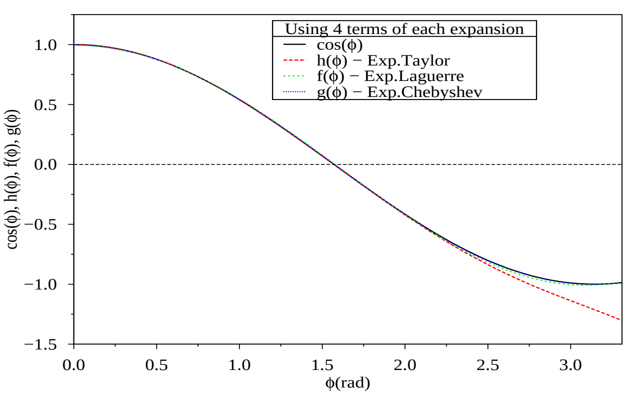

Figure 1: Plot with the results of the expansions of the cosine function using 4 terms by: Taylor series, Chebyshev and Laguerre polynomial expansions.

In Figure 1 we can see that the Laguerre and Chebyshev Polynomial expansions using four terms show a good match of the $\cos(\phi)$ function in the interval $[0, \phi_{max}]$. In the Laguerre polynomial expansion, open parameter λ, with $\lambda = 2/R$, was the one that provides the best match. The Taylor series expansion, using also 4 terms, did not show a good match mainly for the higher angles.

Seismic Modeling

The synthetic velocity model, Figure 2, which represents a complex velocity field with high contrast of velocity and with a salt body, is the first example used to demonstrate the applicability of the seismic modeling method we have proposed using the recursive REM with Laguerre polynomial expansion.

The salt velocity model (Figure 2) has 338 horizontal samples, 210 vertical samples, and for the modeling results shown here we used a time stepping of 0.002 s. The spatial sampling in the velocity grid is 10 m in both directions. A point source, Ricker wavelet, with maximum frequency of 25 Hz, is injected at the coordinates 3000 m in horizontal and 20 m in vertical.
The snapshots were generated at the times 0.4 s, 0.6 s, 0.8 s and 1 s. For comparison, we applied the traditional REM which uses the Chebyshev polynomial expansion and the results are shown in Figure 3. The results obtained using the Laguerre polynomial expansion are presented in Figure 5. The corresponding seismograms (common shot gathers) are shown in Figure 4 and Figure 6, respectively, where the data was recorded at the depth position of 20 m.

These modeling results, obtained for both cases, using the Chebyshev and Laguerre polynomial expansions, for all the snapshots (Figures 3 and 5) and also for both seismograms (Figures 4 and 6), are results free of dispersion noise and attest the efficiency and applicability of the Laguerre polynomial expansion method to approximate the cosine function as well.

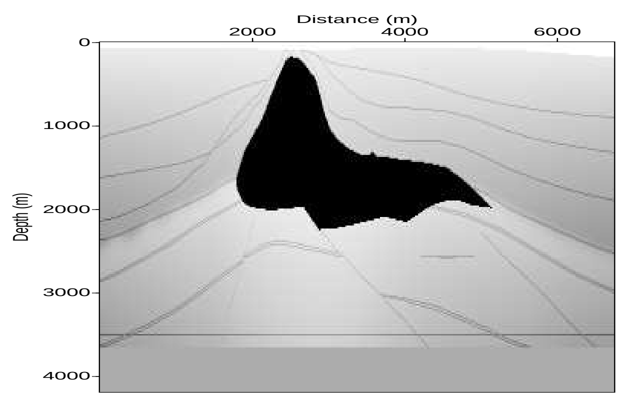

Figure 2: P-velocity model for the salt dome model.

Seismic Migration

Despite of high computational requirement, RTM allows us to migrate reflectors without any dip limitation because it uses a solution based on the full-wave equation (McMechan, 1983; Baysal et al., 1983). This feature, coupled with the inverse propagation over time, instead of depth, makes it possible to deal in complex areas with events such as prismatic waves and also with reflections generated by submarine ascending waves at interfaces characterized by dip angle higher than 90 degrees (Whitmore, 2006). These characteristics justify its efficacy in areas with complex geology, such as those affected by salt tectonic.

The velocity model shown in Figure 7 and its zero-offset section (Figure 8) were used as input data to the RTM algorithm. This model is well known as the SEG-EAGE model. It represents a salt geological structure with a strong velocity contrast with the environment where it is inserted. It has several reflectors and with faults of strong dip, both above and below the salt body. The salt body acts as a disperser of the seismic signal. The velocity model has 1290 horizontal samples, 300 vertical samples and the zero-offset has 2504 time samples per trace with a time sampling interval of $0.002\ s$.

Analysing the migrated results shown on Figures 9, 10 and 11, we can consider that we have achieved a good result in which all complex structures were well imaged: it shows a good imaging of the flank, top and bottom of the salt body. The synclines resulting from deformation caused by salt dome are well mapped, as well as all its contour and the anticline. Comparing the migrated results with the velocity field, we can see that the horizontal line at the bottom of the salt body was very well imaged. The pos-stack migration using the RTM proved that the method proposed here is stable and without dispersion noise. The results presented here were obtained using 5 recursion terms (Figure 9), 10 terms (Figure 10) and 15 terms (Figure 11).

Conclusions

The cosine operator in the two-step wave-equation solution is normally approximated by the Taylor series or by Chebyshev polynomials. In this paper we presented approximations for the cosine operator using the Hermite and Laguerre polynomials. For numerical tests, we implemented the Laguerre polynomial expansion.

The time-stepping evolution of the wave equation using Laguerre polynomials was tested for seismic modeling and migration applications. The results show that the new method can be used satisfactorily for seismic modeling of complex models with high velocity contrast producing snapshots and seismograms stable and free of dispersion noise. The seismic modeling results (snapshots and seismograms) were compared with the results generated by the Chebyshev polynomial expansion and the very good match between the results attests the applicability of the Laguerre polynomial expansion to approximate the cosine operator as well.
The Laguerre polynomial expansion was also tested for reverse time migration (RTM) of seismic data. We applied it for a pos-stack dataset with a complex geologic structure including a salt body and we obtained high quality results where the salt and sub-salt structures were well imaged using different number of recursion terms.

For the numerical tests presented here, the open parameter λ was set equal to $R/8$ and it was the value which ensured best results for modeling and migration. In the Laguerre polynomial expansion, the λ is a free parameter which need to be investigated. More studies is needed to determine which values of λ should be used to ensure goods numerical results.

Acknowledgements

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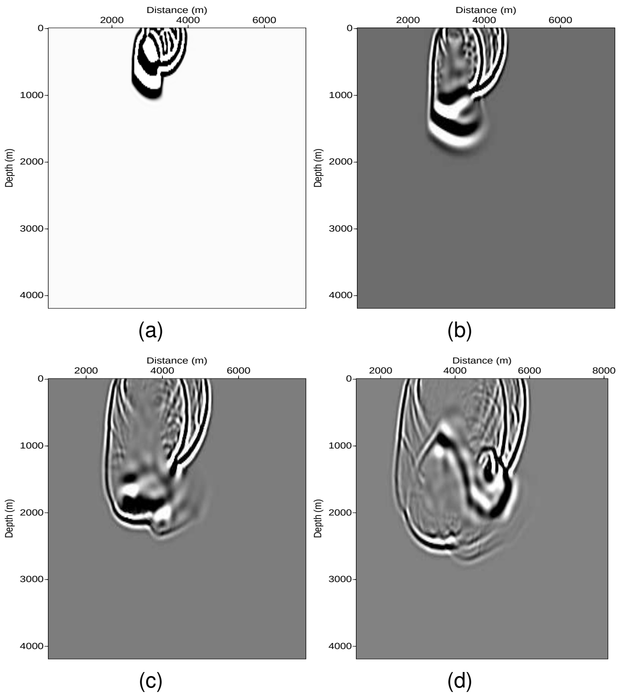

Figure 3: Snapshots generated from the salt dome model (Figure 2), using the Chebyshev expansion at time: 0.4 s (a), 0.6 s (b) 0.8 s (c) and 1 s (d). The source wavelet injected at $x_s = 3000\,m$ and $z_s = 20\,m$, and time stepping of 2 ms.

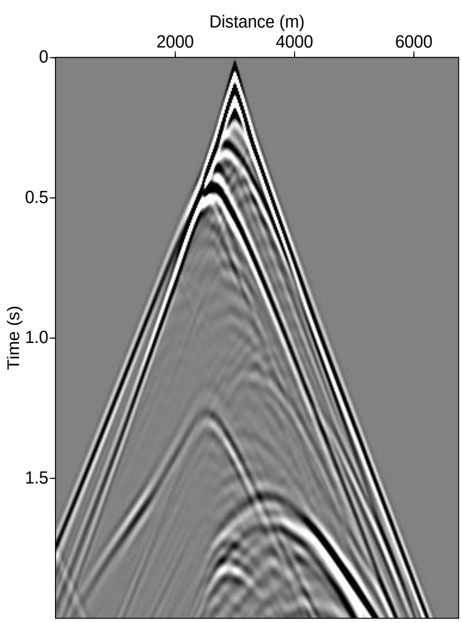

Figure 4: Seismograms generated from the salt dome model using the Chebyshev expansion with time stepping of 2 ms.

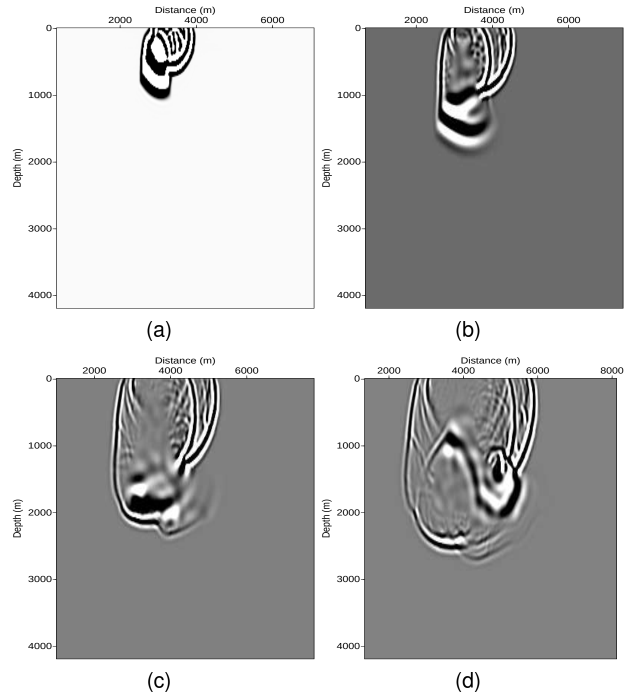

Figure 5: Snapshots generated from the salt dome model using Laguerre expansion at the times: 0.4 s (a), 0.6 s (b) 0.8 s (c) and 1s (d). The source wavelet, injected at $x_s = 3000\,m$ and $z_s = 20\,m$, and time stepping of 2 ms .

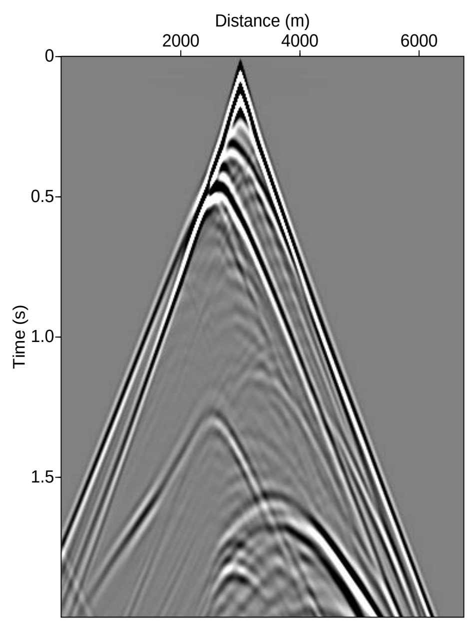

Figure 6: Seismogram generated from the salt dome model using the Laguerre expansion with time stepping of 2 ms.

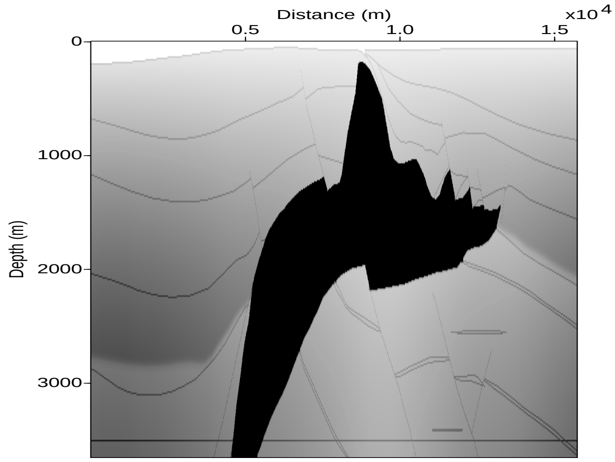

Figure 7: EAGE-SEG velocity model.

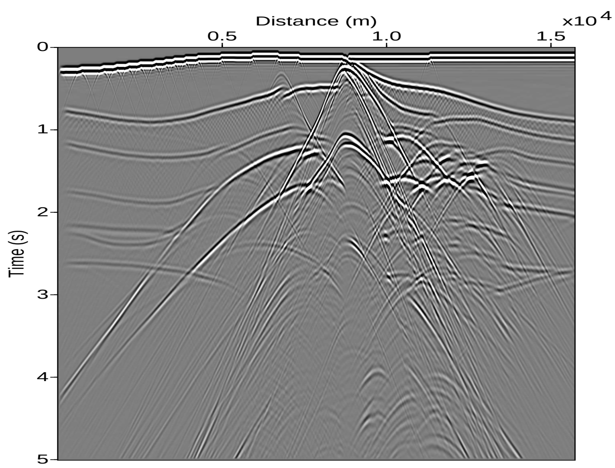

Figure 8: EAGE-SEG zero offset section.

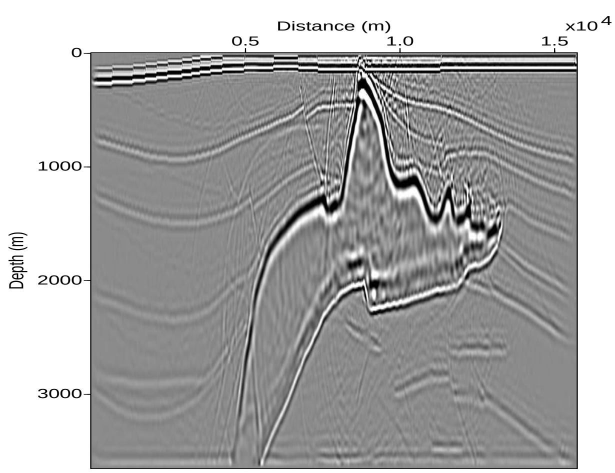

Figure 9: RTM result of the EAGE-SEG dataset using 5 recursion terms

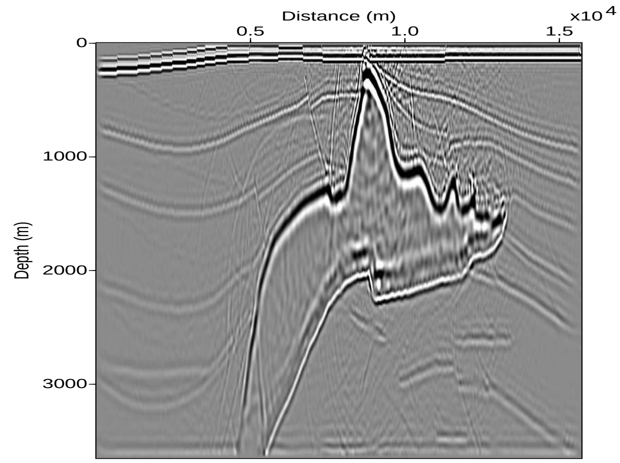

Figure 10: RTM result of the EAGE-SEG dataset using 10 recursion terms.

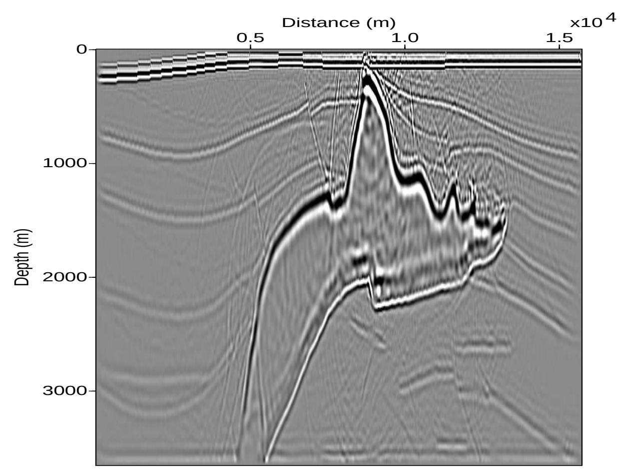

Figure 11: RTM result of the EAGE-SEG dataset using 15 recursion terms